JHEP HYPER VERSION

Received: *June 24, 2000*, Accepted: *August 27, 2000*

Conformal field theories on K3 and three-dimensional gauge theories

Peter Mayr

Theory Division, CERN
1211 Geneva 23, Switzerland
E-mail: Peter.Mayr@cern.ch

Abstract: According to a recent conjecture, the moduli space of the heterotic conformal field theory on a $G \subset$ ADE singularity of an ALE space is equivalent to the moduli space of a pure $\mathcal{N} = 4$ supersymmetric three-dimensional gauge theory with gauge group G. We establish this relation using geometric engineering of heterotic strings and generalize it to theories with non-trivial matter content. A similar equivalence is found between the moduli of heterotic CFT on isolated Calabi-Yau 3-fold singularities and two-dimensional Kazama-Suzuki coset theories.

Keywords: Superstrings and Heterotic Strings, String Duality, Conformal Field Models in String Theory.

JHEP08(2000)042

Contents

1. **Introduction** **1**

2. **Field theory limit** **2**

3. **Dual type-IIA singularities and stable degenerations** **4**

4. **Chains and cycles in $\hat{W}$** **5**

5. **Heterotic CFT on Calabi-Yau 3-fold singularities and Kazama-Suzuki models in two dimensions** **10**

6. **Discussion** **11**

1. Introduction

In a recent paper [1], the exact world-sheet instanton corrected moduli space of the heterotic conformal field theory on a $G = A_1$ singularity of K3 has been determined in the $M_{str} \to \infty$ limit and found to be given by the Atiyah-Hitchin manifold [2]. This is the same moduli as that of the three-dimensional pure $\mathcal{N} = 4$ theory [3] with gauge group $G = A_1$. It has been conjectured in [1] that this relation between the moduli spaces of the conformal field theory and the 3d gauge theories holds for any G. For $G = A_n$, the proposal is verified by the analysis of [4].

The three dimensional $\mathcal{N} = 4$ supersymmetric gauge theories can be studied by a circle compactification of the four dimensional $\mathcal{N} = 2$ theories [3]. The instantons in the three dimensional theory correspond to monopoles and dyons in four dimensions. This situation can also be studied in the field theory limit of a type-II compactification on a Calabi-Yau $W \times S^1$. In this theory, the charged states of the field theory are represented by D-brane wrappings on small p-cycles C_p of a local singularity in W; in the limit $\alpha' \to 0$ with $V(C_p)/(\alpha')^{p/2}$ fixed, most of the fundamental states and gravity decouple and one is left with the light spectrum of the field theory. E.g., in the type-IIB theory, the monopole in four dimensions is a D3-brane wrapped on a small 3-cycle C_3 of a singularity in a local patch of W. The instanton in 3d is the euclidean wrapping of the D3-brane on $C_3 \times S^1$.

The gauge coupling g_3 of the three-dimensional field gets perturbative contributions at tree level and one-loop as well as possible instanton corrections. The instanton correction is of the form $e^{-(I+i\sigma)}$, where σ is the scalar dual to the photon and the instanton action I is approximately $2\pi Mr$ for large r, where M is the mass of the monopole in four dimensions. In the type-II theory the latter is given by the volume of the brane wrapping $V(C_3)$.

The three dimensional gauge theories enjoy often interesting dualities that equate the Coulomb branch of a certain theory with the Higgs branch of another [5, 6, 7]. In the type-II string, this duality is interpreted as a T-duality on the S^1 which takes the type-IIB on $W \times S^1(r)$ to type-IIA on $W \times S^1(1/r)$. If W is a singularity of small 3-cycles as in the previous discussion, the field theory limit of type-IIA describes a theory of hypermultiplets parametrizing the volumes of, and RR fields on, the cycles C_3^i. This is the theory that should be naturally identified with the heterotic CFT in the duality discussed in this paper. The D3 instantons of the type-IIB theory become euclidean D2-branes wrapped on C_3 after the T-duality.

In the heterotic CFT, the expansion is in terms of world-sheet instantons and the corresponding instanton action of the form $e^{-V(c)/\alpha'}$, where c is a 2-cycle in the K3 of the heterotic compactification on which a euclidean fundamental string is wrapped. Moreover tree-level and one-loop of the field theory correspond to the α'^0 and α'^1 contribution to the heterotic string metric, respectively [1]. To obtain the Hyper-Kähler moduli space of the field theory one takes a similar field theory limit of the heterotic string with $\alpha' \to 0$ at fixed $V(c)/\alpha'$. The moduli space of hypermultiplets describes the local deformations of an ADE singularity of an ALE space that governs the local patch of the K3 in this limit.

Although the expansion of the gauge theory and the CFT is morally speaking around the same point, namely in the limit of large instanton action or large volume of the 2-cycles c in the heterotic string or C in the type-IIA theory, respectively, it would be hasty to conclude that the corresponding instanton expansions are exactly the same. In fact it would be surprising if the wrappings of the D2-brane instantons of the type-IIA theory would follow the same rule as that of the fundamental world-sheet instantons of the heterotic CFT. It would be interesting to relate the two instanton expansions.

In this note we derive the conjecture of ref. [1] by using heterotic/type-II duality to show that the type-II geometry W associated to the heterotic string the ADE singularity is the appropriate, known Calabi-Yau singularity of small 3-cycles that yields the G gauge theory upon type-IIB compactification on $W \times S^1$. We will also consider various generalizations of the conjecture.

2. Field theory limit

Consider the heterotic string compactification on $K3 \times T^2$, where the K3 will be replaced by the ADE singularity momentarily. In general this is dual to type-IIA on a K3 fibered Calabi-Yau 3-fold W. If we assume that the bundle factorizes on

K3$\times T^2$ and the heterotic K3 is elliptically fibered, the four-dimensional duality can be pushed up to six dimensions, between F-theory on the, now elliptically fibered, Calabi-Yau W and heterotic string on K3.

Let us first show that the the field theory limit of the heterotic string described by the CFT on the ADE singularity with a Hyper-Kähler moduli space corresponds to a — special — field theory limit of the type-IIA theory The heterotic parameters behave as

$$g_{\text{het}} \to 0, \qquad {\alpha'}_{\text{het}} \to 0, \qquad \frac{V(c)}{{\alpha'}_{\text{het}}} = \text{const.}\,, \tag{2.1}$$

where c stand collectively for a 2-cycle in the ADE singularity.

First note that all p-cycles of K3 except for the cycles c of the singularity grow as $\alpha'^{-p/2}$ in the limit (2.1). This holds in particular for the two universal classes E and B of the elliptically fibered K3 with a section, where E is the class of the fiber and B the class of a section. Consider the singularity from a collision of singular elliptic fibers at a point s_0 of B. The transverse dimensions to the singularity are described by E and B, which become non-compact in the above limit. Therefore we are left with the local singularity of a non-compact ALE space.

The four-dimensional compactification above can be considered as a compactification of the six-dimensional duality between type-IIA on K3 and heterotic string on T^4 on a base $\mathbf{P}^1$ denoted by B. From the relations in [8] one can then easily see that the four-dimensional coupling constants are related by

$$\frac{1}{g^2_{4,het}} \sim \frac{V_{\text{het}}(B)}{g^2_{6,het}} \sim V_{II}(B), \qquad \frac{1}{g^2_{4,II}} \sim \frac{V_{II}(B)}{g^2_{6,II}} \sim V_{\text{het}}(B). \tag{2.2}$$

Note that B is the base of the elliptically fibered K3 of the heterotic theory and the K3 fibration of W in the type-IIA theory, respectively. Since $V_{\text{het}}(B)$ becomes large, we have a weakly coupled type-IIA theory. To have a weakly coupled heterotic string, $V_{II}(B)$ becomes large, too.

Moreover the type-IIA string is obtained from wrapping a heterotic 5-brane on T^2 times the elliptic fiber of K3. For generic volume of T^2 this 4-cycle becomes very large and therefore $\alpha' \to 0$ in the type-II theory. However as we will show in the next section, the limit we have to take is not the naive $\alpha' \to 0$ limit: to the small 2-cycles c_i in K3 correspond small 3-cycles C_i in W with a fixed volume $V(C_i)/\alpha'^{3/2}_{II}$.

Above we have recovered the definition of the field theory limit of the dual type-II string studied in the geometric engineering approach of type-II strings [9] on Calabi-Yau singularities. The fact that it is the fiber of the heterotic K3 Z that gets large (and not just the 4-cycle of the 5-brane wrapping) implies a special limit in the complex structure of the Calabi-Yau singularity. We will show in the next section that the special singularities one obtains in this way are precisely those used in [10] to extend the geometric engineering approach of [9] to describe heterotic moduli spaces by F-theory limits of Calabi-Yau singularities. The subsequent sections may be understood without going through the mathematics of the next section.

3. Dual type-IIA singularities and stable degenerations

A second consequence of the large elliptic fiber of Z is a limit in the complex structure of W that is known as a stable degeneration [11, 12, 13]. In particular it was proposed in [11] that if W is a K3 manifold, one can see the heterotic T^2 in W on the nose in the limit of the stable degeneration, which in heterotic terms corresponds to a very large T^2. Since we are interested in the complex structure of W parametrized by the hypermultiplets of the type-IIA theory, we can consider F-theory compactification on W dual to heterotic on Z without the extra T^2. The type-IIA theory is related to F-theory by the extra T^2 compactification which will affect only the Kähler moduli space of the Calabi-Yau W.

In the following we will describe in some detail general deformations of the stable degeneration and make contact with the field theory limit of the type-IIA on non-compact singularities considered in [10]. We will verify the identifications made in [11] and add also the information about the bundles on the heterotic manifold Z in this way. This will serve as a starting point for the identification of the type-II singularity dual to the ADE singularity of the heterotic string in the next section.

The elliptically fibered manifold W for F-theory compactification in Weierstrass form is given by

$$p_W = y^2 + x^3 + xf + g = 0\,, \tag{3.1}$$

with f and g some functions depending on the coordinates on the base B of the elliptic fibration $\pi : W \to B$. To describe the limit in complex structure we are interested in, we consider families of the F-theory manifold W in a fibration over the complex plane, similarly as in [13]. Let us write (3.1) in more detail as

$$p_W = y^2 + x^3 + x\sum_\alpha s^{4-\alpha}\tilde{s}^{4+\alpha}f_\alpha^{(4)} + \sum_\alpha s^{6-\alpha}\tilde{s}^{6+\alpha}f_\alpha^{(6)}\,, \tag{3.2}$$

where $f_\alpha^{(k)}$ are constants if W is a K3, but more generally functions on the $n-1$ dimensional base B' of the K3 fibration $\pi' :\ W \to B'$ of the Calabi-Yau $n+1$-fold W. Moreover s and $\tilde{s}$ are homogeneous coordinates on the base $\mathbf{P}^1$ of the elliptic fibration of the K3 fiber.

To perform the limit, we consider families Y of W in a fibration over another $\mathbf{P}^1$ parametrized by μ

$$p_Y = y^2 + x^3 + x\sum_{\alpha,\beta} s^{4-\alpha}\tilde{s}^{4+\alpha}\mu^{4-\beta}f_\alpha^{(4)}a_{\alpha,\beta}^{(4)} + \sum_{\alpha,\beta} s^{6-\alpha}\tilde{s}^{6+\alpha}\mu^{6-\beta}f_\alpha^{(6)}a_{\alpha,\beta}^{(6)}\,,$$

where $a_{\alpha,\beta}^{(k)}$ are some complex constants that we will set to 0 or 1 in the following. In general, the fiber of Y at $\mu=0$ is a (certain deformation) of the original manifold W. However if we restrict the complex structure of Y such that $a_{\alpha,\beta}^{(k)}=0$ for $\alpha+\beta>k$

the fiber at $\mu = 0$ becomes

$$p_Y = y^2 + x^3 + xs^4 \sum_{\alpha \leq 0} s^{-\alpha} \tilde{s}^{4+\alpha} f_\alpha^{(4)} + s^6 \sum_{\alpha \leq 0} s^{-\alpha} \tilde{s}^{6+\alpha} f_\alpha^{(6)} ,$$

which has a non-minimal singularity at $y = x = s = 0$. After a blow up of Y,

$$y = \rho^3 y,\ x = \rho^2 x,\ s = \rho s,\ \mu = \rho\mu ,$$

we arrive at

$$p_{Y'} = y^2 + x^3 + x \sum_{\alpha,\beta} s^{4-\alpha} \tilde{s}^{4+\alpha} \mu^{4-\beta} \rho^{4-\alpha-\beta} f_\alpha^{(4)} + \sum_{\alpha,\beta} s^{6-\alpha} \tilde{s}^{6+\alpha} \mu^{6-\beta} \rho^{6-\alpha-\beta} f_\alpha^{(6)} .$$

The singular fiber at $\mu = 0$ is now replaced by a reducible fiber $\hat{W}$ with two components corresponding to $\mu = 0$ and $\rho = 0$:

$$\begin{aligned} p_{W_1} = p_0 + p_+ = &\left(y^2 + x^3 + x f_0^{(4)} + f_0^{(6)} \right) + \\ &+ \left(x \sum_{\alpha<0} \tilde{s}^{4+\alpha} \rho^{-\alpha} f_\alpha^{(4)} + \sum_{\alpha<0} \tilde{s}^{6+\alpha} \rho^{-\alpha} f_\alpha^{(6)} \right) , \\ p_{W_2} = p_0 + p_- = &\left(y^2 + x^3 + x f_0^{(4)} + f_0^{(6)} \right) + \\ &+ \left(x \sum_{\alpha>0} s^{4-\alpha} \mu^{\alpha} f_\alpha^{(4)} + \sum_{\alpha>0} s^{6-\alpha} \mu^{\alpha} f_\alpha^{(6)} \right) . \end{aligned} \tag{3.3}$$

The two components W_1 and W_2 intersect over the locus $\mu = \rho = 0$ described by $p_0 = 0$. According to [11] this intersection is to be identified with the heterotic manifold Z.

This proposal can be made precise using local mirror symmetry which allows to also extract the heterotic bundle [10]. As shown there, a type-IIA compactification on certain non-compact Calabi-Yau singularities $\mathcal{W}$ results in a gravity free field theory with the moduli space of holomorphic stable bundles on a compact Calabi-Yau Z_n defined as a certain hypersurface in $\mathcal{W}$. For appropriate structure group of the bundle this data can be interpreted as a heterotic compactification in the point particle limit. From comparison with (3.3), one observes that if we consider separately a manifold $\mathcal{W}_1$ described by $p_{W_1} = 0$, it describes precisely a non-compact Calabi-Yau singularity of the type considered in [10]. The singularity $\mathcal{W}_1$ indeed describes the moduli space of holomorphic stable bundles V on $Z : \ p_0 = 0$. Moreover we can use the identifications for the singularity $\mathcal{W}_1$ derived in [10] to read off the moduli of the bundle V on Z.

4. Chains and cycles in $\hat{W}$

Let us show now how the vanishing 2 cycles c_i of the heterotic ADE singularity correspond to shrinking 3-cycles in the dual type-II manifold $\hat{W}$. The latter support

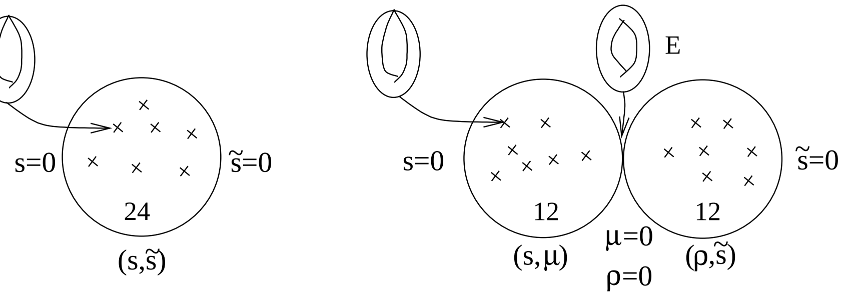

Figure 1: The original K3 and the generic stable deformation.

the D2 instantons of the field theory limit of the type-IIA theory and the monopole wrappings of the T-dual type-IIB theory that represents the Coulomb branch of the three-dimensional gauge theory.

Let us start with the illustrative and simpler case of F-theory on a K3 W dual to heterotic string on T^2. It shares many of the relevant aspects with the case of the heterotic string on K3 that will be obtained by fibering over an extra $\mathbf{P}^1$.

Before the stable degeneration, W is an elliptic fibration over a $\mathbf{P}^1$ with 24 singular elliptic fibers located at the zeros of the discriminant $\Delta = 4f^3 + 27g^2$, where f and g are degree 8 and 12 polynomials in the variable s on the base $\mathbf{P}^1$, respectively. After the stable degeneration, W has split into the two components W_1 and W_2 described by eq. (3.3), intersecting over the elliptic curve $E:\ p_0 = 0$. The base of the elliptic fibrations of the elliptic rational surfaces W_i are two $\mathbf{P}^1$'s that intersect over a point, $\mu = \rho = 0$. There are 12 points on each $\mathbf{P}^1$ above which the elliptic fibers becomes singular (see figure 1).

There are two basic degenerations of this elliptic fibration. First we can bring together singular elliptic fibers of a single component, say W_1. This introduces a singularity of small 2-cycles in W_1 with a type determined by the classification of Kodaira. Since the relative location of the singular elliptic fibers is described by p_+ in eq. (3.3), with parameters that translates to that of the heterotic bundle as described in [10], this corresponds to a degeneration of the gauge bundle in the heterotic string on T^2.

The second kind of degeneration consists of bringing together a singular fiber from W_1 with another singular fiber from W_2. This is only possible at the intersection of the two base $\mathbf{P}^1$'s at $\mu = \rho = 0$. Again we obtain a shrinking 2-cycle c in $\hat{W}$. However note that c intersects E along a 1-cycle γ.

To see the cycle c and the fact that its volume vanishes if and only if the 1-cycle γ shrinks, consider the local situation in W_1 described by choosing

$$f = f_0 + \rho f_1 = (-3a^2) + \rho f_1\,, \qquad g = g_0 + \rho g_1 = (2a^3 - \epsilon) + \rho g_1\,, \tag{4.1}$$

in (3.1). For small ϵ, the heterotic torus at $\rho = 0$ has a small $W \times S^1$ described by

$$\gamma:\ y^2 + x^2 = \epsilon\,.$$

Away from $\rho = 0$ there is still an $W \times S^1$, $\gamma(\rho)$, with a radius $\sqrt{\epsilon(\rho)}$ depending on the value of ρ. From the vanishing of the discriminant $\Delta = \Delta_0 + \rho\Delta_1$ we see that $\gamma(\rho)$ shrinks to zero size at a point $\rho_0 = -\Delta_0/\Delta_1$. We can therefore define a two 2-chain $K^{(1)}$ in W_1 from gluing together 1-cycles $\gamma(\rho)$ along the interval $\rho = [0, \rho_0]$.

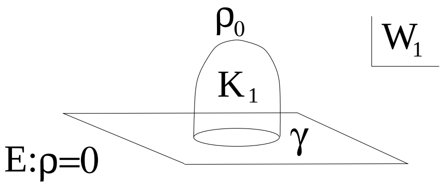

Figure 2: The 2-chain K^1 in W_1.

Note that the cycle $\gamma = \gamma(0)$ in E shrinks iff $\Delta_0 = 0$, which implies $\rho_0 \to 0$. Thus precisely if γ shrinks, $K^{(1)}$ shrinks, too. In fact the volume of the 2-chain $K^{(1)}$ is determined by the integral

$$\int_{K^{(1)}} \Omega \sim \int_{K^{(1)}} \frac{dx\ d\rho}{y} = \frac{\pi\epsilon}{\sqrt{3a}(af_1 + g_1)}\,, \tag{4.2}$$

where the proportionality constant depends on the normalization of the 2-form Ω. In the total manifold $\hat{W}$ we obtain a vanishing 2-cycle from the two vanishing 2 chains $K^{(\alpha)} \subset W_\alpha$ joining along the vanishing cycle γ. Note that we can write the discriminant of $\hat{W}$ as $\Delta(\hat{W}) = \Delta(W_1) \cdot \Delta(W_2)$, which vanishes to second order at $\mu = \rho = 0$ due to the first order vanishing of $\Delta(W_i)$.

Thus this second kind of degeneration is related to a geometric degeneration of the heterotic torus E. This does not make too much sense in the case of the torus, since its degenerations are very limited and in fact at infinite distance in the moduli space.[1] We can also combine the two kind of degenerations and bring together several singular fibers from W_1 and W_2 at the intersection. From the above this should correspond to a degeneration of the bundle on top of the singular heterotic geometry.

Having discussed this toy example, let us consider the case of heterotic string on a non-trivial Calabi-Yau Z_n, $n > 1$, in particular for $Z = K3$. The new ingredient is that we can now have geometric singularities of Z at finite distance in the moduli, in particular ADE singularities of ALE spaces. Let us fiber the eight-dimensional picture in figure 1 over an extra $\mathbf{P}^1$, parametrized by t. The local structure of the 3-fold geometry $\hat{W}$ for small 2-cycles c_i in Z is similar as in eq. (4.1) with an extra dependence on the base $\mathbf{P}^1$:

$$f = f_0(t) + \rho\ f_1(t), \qquad g = g_0(t) + \rho\ g_1(t). \tag{4.3}$$

Similarly we can split the discriminant of the elliptic fibration as $\Delta = \Delta_0(t) + \rho\ \Delta_1(t)$. Near a G singularity of the K3, the zeros of Δ_0 are described by

$$\Delta_0 = \prod_i^\delta (t - t_i(a_k)) = t^\delta + \cdots\,, \tag{4.4}$$

[1] However note that in this limit we have $U = \infty = T$ so we can interpreted the SU(2) gauge symmetry of the type-IIA string on the vanishing 2-cycle as the heterotic gauge symmetry enhancement at $T = U$ in the limit $T \to \infty$.

where $a_k,\ k = 1,\ldots,\mathrm{rk}\ G$ denote the deformations of the singularity and δ is a certain integer that counts the number of simple singular fibers needed to produce the G singularity in the elliptic fibration.[2] Classically, the G singularity is recovered in the limit, where the perturbations a_k become zero.

Moving away from $\rho = 0$ to describe the 3-fold $\hat{W}$, the zeros t_i of the discriminant Δ depend on ρ:

$$\Delta = \Delta_0 + \rho\ \Delta_1 = \prod_i^{\delta} (t - t_i(a_k, \rho))\,. \tag{4.5}$$

Note that for each pair of $t_i(a_k, \rho)$, we can define a 2-cycle $c_{ij}(\rho)$ by stringing together 1-cycles in E along the interval $[t_i(a_k, \rho), t_j(a_k, \rho)]$, very similar as we did in the construction of the 2-chains $K^{(\alpha)}$. Moreover the 2-cycle $c_{ij}(\rho)$ attaches to a 2-cycle c_{ij} of the K3 Z for $\rho = 0$. In addition it shrinks to zero size at a point $\rho = \rho_{ij}$ where

$$t_i(a_k, \rho_{ij}) = t_j(a_k, \rho_{ij}). \tag{4.6}$$

Very similar as in the previous case we can now build 3-chains $K_{ij}^{(\alpha)}$ by gluing together the 2-spheres $c_{ij}(\rho)$ along the interval $[0, \rho_{ij}]$.

Finally note that for very similar reasons as for the 2-chains constructed before, the volume of the 3-chain $K_{ij}^{(\alpha)}$ vanishes precisely if the 2-cycle c_{ij} shrinks and in particular all 3-chains shrink near the singularity of the heterotic K3 Z

$$a_k \to 0 \ \Rightarrow\ \rho_{ij} \to 0\,. \tag{4.7}$$

One can calculate also the volume of the individual 3-chains $K_{ij}^{(\alpha)}$ similarly as in (4.2).

We have therefore succeeded to establish a 1-1 correspondence between the small 2-cycles c_{ij} of the resolved ADE singularity and small 3-cycles $C_{ij} = K_{ij}^{(1)} \cup K_{ij}^{(2)}$ of a singularity reached in the complex structure of the type-II compactification manifold $\hat{W}$. The lattice $H_3(\hat{W})$ inherits the ADE structure from the lattice $H_2(Z)$. Note that the cycles C_{ij} intersect each other according to the ADE Dynkin diagram. This fits with the physics expectations that the instanton corrections to the three-dimensional gauge theory represented by type-IIB on $\hat{W}$ arise from the monopoles and dyons, as opposed to the wrappings of purely electric cycles.

The singularity of the type-II manifold $\hat{W}$ is precisely the one which is known to produce the G gauge theory in three dimensions in a type-IIB compactification on $\hat{W} \times S^1$ [9]. Note that we needed the extra S^1 to translate from the heterotic type-IIA dual that describes the "Higgs branch" of hypermultiplets, to the Coulomb branch of the G gauge theory of the type-IIB field theory limit, which corresponds to the dual theory in the sense of [5, 6]. Let us also identify the states dual to the monopoles

[2] E.g. $\delta = N$ for $\mathrm{SU}(N)$ and more generally $\delta = \mathrm{rk}G + 1 + \nu$, where ν is the number of trivalent vertices in the Dynkin diagram of G. See ref. [14].

from D3-branes wrapped on C_{ij}. In the T-dual type IIA theory they become D4-branes wrapped on $C_{ij} \times S^1$. In the heterotic theory they become 5-branes wrapped on $c_{ij} \times T^3$.

Let us consider now possible generalizations of the above picture by moving additional singular elliptic fibers into the heterotic singularity. As discussed in the eight-dimensional picture, if we let collide fibers at some point $\rho = \rho'$ of the base of the elliptic surface W_i, this corresponds to a degeneration of the heterotic bundle. Moving this point into the ADE singularity, $\rho' \to 0$, we expect that in general the gauge bundle is not trivial on the singularity. This would correspond to theories with non-perturbative heterotic dynamics, such as the extra gauge symmetry enhancement discussed in [10].

Let us first consider a simpler case, where it is easier to find the answer. In the six-dimensional compactification we can also consider moving in additional singular elliptic fibers into the singularity using the t direction. In this way we can keep the singularity of the elliptic surface W_i, and thus the bundle, fixed on the singularity. In particular, if we choose a totally trivial gauge bundle there are 24 small instantons located at points $t = t_i$ of the base of Z. The case where some of these positions coincide with the location of the ADE singularity has been studied in great detail in [15, 13]. With some effort one can show that after moving in such points there are new 3-cycles in $\hat{W}$ that correspond to adding matter in the gauge theory following the rules of geometric engineering of type-II strings. Note that we can only get matter that can be obtained from adjoint breaking of a higher rank gauge group [16] in this way. We will take a shortcut here that uses the results on three-dimensional mirror symmetry described in [6, 7], which will lead us also to a natural conjecture for new three-dimensional dualities. Specifically, ref. [7] established a series of three-dimensional dualities using geometric engineering of type-II on geometric singularities, which is precisely the situation that we consider. It was shown that if the type-IIB theory on a Calabi-Yau W describes a $\mathrm{U}(N)$ theory with $N_f = k$ fundamentals, than the type-IIA theory on the same manifold describes the Higgs branch of a dual theory with gauge group $\mathrm{U}(1) \cdot \mathrm{U}(2) \cdots \mathrm{U}(N-1) \cdot \mathrm{U}(N)^{k-2N+1} \cdot \mathrm{U}(N-1) \cdots \mathrm{U}(2) \cdot \mathrm{U}(1)$. On the other hand [15, 13] this is precisely the gauge theory of k small instantons on a $\mathrm{SU}(N)$ singularity of K3! Thus we see that by considering the field theory limit of the heterotic theory with small instantons on the K3 singularity it essentially follows from the results in [7] that the heterotic moduli of the A_{N-1} singularity with k instantons is the same as that of the three-dimensional $\mathrm{SU}(N)$ theory with $N_f = k$.[3]

[3]It is worth noticing that this result is consistent with a proposal for mirror symmetry of F-theory formulated in [17] and derived and refined in [10]. In particular it states that if F-theory on a 3-fold $\tilde{W}$ describes the Coulomb branch of k small heterotic instantons on a G singularity of K3, than the mirror $\tilde{W}^*$ is dual to a heterotic theory on the Higgs branch describing a large G instanton with instanton number k. After a T^3 compactification, the two theories are related by the three-dimensional duality as above.

It is natural to wonder whether this relation is true more generally, also in those cases, where three-dimensional dualities have not yet been established or even formulated. The prediction would be that the theory dual to the three-dimensional gauge theory with gauge group G and k matter multiplets is the gauge theory with gauge group $\tilde{G}$, where $\tilde{G}$ is the extra local gauge group that appears if k small heterotic instantons collide with a G singularity of K3.

Let us now come back to the case where we move singular fibers of the elliptic surface W_1 into the ADE singularity. We have in the moment no good understanding of what the three-dimensional theories associated to such a degeneration may be, however there is also a plausible candidate for this case from the geometric engineering of type-II strings. Note first that if we take the base to be $\mathbf{P}^1 \times \mathbf{P}^1$, we can interpret any of the two $\mathbf{P}^1$'s as the base of the heterotic K3. Accordingly we have two heterotic theories that are related by heterotic-heterotic duality [18]. An extension of this duality for singular K3's and bundle singularities exists and has been described in [10]. In the previous case we have considered having a G singularity in the K3 with base $\mathbf{P}^1$ parametrized by t. Of course nothing changes if we consider instead only a singularity G' in the heterotic theory with base the other $\mathbf{P}^1(s, \tilde{s})$. If we now consider a manifold where the G and G' singularities collide, we have in some sense a collision of two gauge theories with groups G and G'. This reminds very much of the structure of the conformal field theories engineered from type-II strings in [19]. In this case one has a gauge group $G = \prod \mathrm{SU}(k\ n_i)$, with n_i the Dynkin indices of a gauge group G' and bi-fundamentals in each group factors connected by a link in the Dynkin diagram. The coupling constants of the gauge theory are in fact related to the Coulomb parameter of a G' gauge theory which is however broken to a trivial $\mathrm{U}(1)^{\mathrm{rk}\, G'}$ factor in the region where we observe the G gauge theory. However it is also possible to to move towards the origin of the Coulomb branch of G and G' at the same time. One obtains in this limit an exotic gravity free theory with string like degrees of freedom [19]. The compactification of this theory on S^1 seems to be a good candidate for the three-dimensional theories related to a collision of a singularity in W_1 with an ADE singularity in Z.

5. Heterotic CFT on Calabi-Yau 3-fold singularities and Kazama-Suzuki models in two dimensions

By the same arguments as in the previous section we can construct more generally a correspondence between $m+1$-chains in the stable degeneration of a Calabi-Yau $m+1$-fold W_{m+1} and m cycles in the intersection of its two components which is an elliptic m-dimensional Calabi-Yau manifold. For the case $m = 3$ we obtain a correspondence between the heterotic CFT on the isolated singularity

$$y^2 + H(x, s, t) = 0\,, \tag{5.1}$$

and type-II strings on a Calabi-Yau 4-fold with a corresponding singularity of one dimension higher described locally by a similar equation with H replaced by $H + w^2$. Here $H = 0$ describes the ADE surface singularities

$$\begin{aligned} H &= s^n + t^2 + x^2 \quad A_{n-1}\,, & H &= s^3 + st^3 + x^2 \quad E_7\,, \\ H &= s^n + st^2 + x^2 \quad D_{n+1}\,, & H &= s^3 + t^5 + x^2 \quad E_8\,, \\ H &= s^n + t^4 + x^2 \quad E_6\,. \end{aligned} \tag{5.2}$$

The field theory limit of type-IIA theory on these singularities has been identified in [20] as a Kazama-Suzuki model based on a coset that depends on the choice of a RR background on 4-cycles. The RR fields of type-IIA map to gauge fields of the hetrotic string. We conclude that the heterotic CFT on the 3-fold singularity with appropriate gauge field background is on the same moduli as the corresponding Kazama-Suzuki model. A simple consistency check on this equivalence is the following one. Using linear sigma models it was argued in [1] that the heterotic CFT should be smooth on a singular Calabi-Yau n-fold for any n as long as the gauge fields are trivial on the singularity. For trivial RR-fields one obtains a massive Kazama-Suzuki model with no light degrees of freedom. This matches the non-singular behavior of the heterotic theory advocated in [1]. It would be interesting to compare in detail the moduli spaces for the more interesting case with non-trivial background fields, in particular when the heterotic string is compactified on the $(2,2)$ theory.

6. Discussion

The equivalence of the moduli space of the heterotic CFT on ADE singularities and the moduli space of three dimensional gauge theories, or the dual monopole moduli spaces, is an interesting tool to study several aspects. E.g., the metric of the Atiyah-Hitchin manifold is known [2] and one can extract the world-sheet instanton expansion of the heterotic CFT, similarly as one obtains the world-sheet instantons of the type-II string from mirror symmetry. By an appropriate reparametrization, we expect to reproduce the D2 instanton expansion of the type-IIA string on $\hat{W}$. This allows to analyze a new class of brane wrappings. It is interesting to note in this context that it is easy to verify that the Atiyah-Hitchin metric is compatible with the claim in [3] that in the gauge theory variables there is only a single instanton contribution; this is a first non-trivial compatibility check. Reversing the logic it would also be very interesting to study metrics on monopole moduli spaces from the CFT approach. Similarly it would be interesting to study the relation between instanton corrections to the metric of the heterotic 3-fold and the two-dimensional Kazama-Suzuki models.

Note added: While preparing this note for publication, two related papers appeared on the subject. Ref. [21] gives an independent derivation of the conjecture of ref. [1] using M-theory. Ref. [22] treats the special case $G = A_1$ case, already analyzed in [1], using a "new" variant of geometric engineering for heterotic strings. In fact this is the same geometric engineering used in this paper[4] and derived and studied in detail by local mirror symmetry in [10, 19, 9].

Acknowledgments

We would like to thank W. Lerche and Y. Oz for valuable discussions.

References

[1] E. Witten, *Heterotic string conformal field theory and A-D-E singularities*, hep-th/9909229.

[2] M.F. Atiyah and N. Hitchin, *The geometry and dynamics of magnetic monopoles*, Princeton University Press, 1988.

[3] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three dimensions*, hep-th/9607163.

[4] A. Sen, *Dynamics of multiple Kaluza-Klein monopoles in M and string theory*, *Adv. Theor. Math. Phys.* **1** (1998) 115 [hep-th/9707042].

[5] K. Intriligator and N. Seiberg, *Mirror symmetry in three dimensional gauge theories*, *Phys. Lett.* **B 387** (1996) 513 [hep-th/9607207].

[6] J. de Boer, K. Hori, H. Ooguri and Y. Oz, *Mirror symmetry in three-dimensional gauge theories, quivers and D-branes*, *Nucl. Phys.* **B 493** (1997) 101 [hep-th/9611063];
M. Porrati and A. Zaffaroni, *M-theory origin of mirror symmetry in three dimensional gauge theories*, *Nucl. Phys.* **B 490** (1997) 107 [hep-th/9611201].

[7] K. Hori, H. Ooguri and C. Vafa, *Non-abelian conifold transitions and $N = 4$ dualities in three dimensions*, *Nucl. Phys.* **B 504** (1997) 147 [hep-th/9705220].

[8] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124].

[9] For an introduction and background material see P. Mayr, *N=1 Heterotic string vacua from mirror symmetry*, hep-th/9904115; *Non-perturbative $N = 1$ strings from geometric singularities*, *Class. and Quant. Grav.* **17** (2000) 1205 [hep-th/9910216]; *D-branes and F-theory*, lectures the Trieste Spring School 1999, to appear in the proceedings.

[4] In particular the definition of the geometric limit in [22, pag. 6] is a special example of the analysis in [10, appendix A].

[10] P. Berglund and P. Mayr, *Heterotic string/F-theory duality from mirror symmetry*, *Adv. Theor. Math. Phys.* **2** (1999) 1307 [hep-th/9811217].

[11] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds, 2*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161];
Compactifications of F-theory on Calabi-Yau threefolds, 1, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114].

[12] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Comm. Math. Phys.* **187** (1997) 679 [hep-th/9701162].

[13] P.S. Aspinwall and D.R. Morrison, *Point-like instantons on K3 orbifolds*, *Nucl. Phys.* **B 503** (1997) 533 [hep-th/9705104].

[14] M. Bershadsky et al., *Geometric singularities and enhanced gauge symmetries*, *Nucl. Phys.* **B 481** (1996) 215 [hep-th/9605200].

[15] K. Intriligator, *RG fixed points in six dimensions via branes at orbifold singularities*, *Nucl. Phys.* **B 496** (1997) 177 [hep-th/9702038];
J.D. Blum and K. Intriligator, *New phases of string theory and 6d RG fixed points via branes at orbifold singularities*, *Nucl. Phys.* **B 506** (1997) 199 [hep-th/9705044];
Consistency conditions for branes at orbifold singularities, *Nucl. Phys.* **B 506** (1997) 223 [hep-th/9705030].

[16] S. Katz and C. Vafa, *Matter from geometry*, *Nucl. Phys.* **B 497** (1997) 146 [hep-th/9606086].

[17] E. Perevalov and G. Rajesh, *Mirror symmetry via deformation of bundles on K3 surfaces*, *Phys. Rev. Lett.* **79** (1997) 2931 [hep-th/9706005].

[18] M.J. Duff, R. Minasian and E. Witten, *Evidence for heterotic/heterotic duality*, *Nucl. Phys.* **B 465** (1996) 413 [hep-th/9601036];
D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds, 2*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161]; *Compactifications of F-theory on Calabi-Yau threefolds, 1*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114].

[19] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of 4d $N = 2$ gauge theories, 1*, *Adv. Theor. Math. Phys.* **1** (1998) 53 [hep-th/9706110].

[20] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, *Nucl. Phys.* **B 584** (2000) 69 [hep-th/9906070].

[21] M. Rozali, *Hypermultiplet moduli space and three dimensional gauge theories*, *J. High Energy Phys.* **12** (1999) 013 [hep-th/9910238].

[22] P.S. Aspinwall and M.R. Plesser, *Heterotic string corrections from the dual type-II string*, *J. High Energy Phys.* **04** (2000) 025 [hep-th/9910248].

JHEP08(2000)042